The Factors of Life

The Factor Theorem and Remainder Theorem

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LEARNING GOALS

In this lesson, you will:

- Use the Remainder Theorem to evaluate polynomial equations and functions.
- Use the Factor Theorem to determine if a polynomial is a factor of another polynomial.
- Use the Factor Theorem to calculate factors of polynomial equations and functions.

KEY TERMS

- Remainder Theorem
- Factor Theorem

When you hear the word remainder, what do you think of? Leftovers? Fragments? Remnants?

The United States, as a country, produces a great deal of their own "leftovers." The amount of paper product leftovers per year is enough to heat 50,000,000 homes for 20 years. The average household disposes of over 13,000 pieces of paper each year, most coming from the mail. Some studies show that 2,500,000 plastic bottles are used every hour, most being thrown away; while 80,000,000,000 aluminum soda cans are used every year. Aluminum cans, that have been disposed of and not recycled will still be cans 500 years from now.

There are certain things you can do to help minimize the amount of leftovers you produce. For example, recycling one aluminum can save enough energy to watch TV for three hours. Cans that are recycled, can be good as new, and resold as fast as 60 days from when it was recycled. If just $\frac{1}{10}$ of the daily newspapers were recycled, 25,000,000 trees could be saved per year. Recycling plastic uses half the amount of energy it would take to burn it.

PROBLEM 1 You Have the Right to the Remainder Theorem

You learned that the process of dividing polynomials is similar to the process of dividing integers. Sometimes when you divide two integers there is a remainder, and sometimes there is not a remainder. What does each case mean? In this lesson, you will investigate what the remainder means in terms of polynomial division.

Remember from your experiences with division that:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

or

$$\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}.$$

It follows that any polynomial, $p(x)$, can be written in the form:

$$\frac{p(x)}{\text{linear factor}} = \text{quotient} + \frac{\text{remainder}}{\text{linear factor}}$$

or

$$p(x) = (\text{linear factor})(\text{quotient}) + \text{remainder}.$$

Generally, the linear factor is represented by the form $(x - r)$, the quotient is represented by $q(x)$, and the remainder is represented by R, meaning:

$$p(x) = (x - r)q(x) + R.$$

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1. Given $p(x) = x^3 + 8x - 2$ and $\frac{p(x)}{(x - 3)} = x^2 + 3x + 17$ R 49.
   a. Verify $p(x) = (x - r)q(x) + R$.

   b. Given $x - 3$ as the linear factor, evaluate $p(3)$.

2. Given $p(x) = (x - r)q(x) + R$ calculate $p(r)$.

3. Explain why $p(r)$, where $(x - r)$ is a linear factor, will always equal the remainder R, regardless of the value of the quotient factor.

4. What conclusion can you make about any polynomial evaluated at r?

The **Remainder Theorem** states that when any polynomial equation or function, $f(x)$, is divided by a linear factor $(x - r)$, the remainder is $R = f(r)$ or the value of the equation or function when $x = r$.

5. Given $p(x) = x^3 + 6x^2 + 5x - 12$ and $\frac{p(x)}{(x - 2)} = x^2 + 8x + 21$ R 30,

Rico says that $p(-2) = 30$ and Paloma says that $p(2) = 30$.

Without performing any calculations, who is correct? Explain your reasoning.

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6. The function, $f(x) = 4x^2 + 2x + 9$ generates the same remainder when divided by $(x - r)$ and $(x - 2r)$ when r is not equal to 0. Calculate the value(s) of r.

PROBLEM 2 Factors to Consider

Consider the factors of 24: 1, 2, 3, 4, 6, 8, 12, 24.

Notice that when you divide 24 by any of its factors the remainder is 0. This same principle holds true for polynomial division.

The **Factor Theorem** states that a polynomial has a linear polynomial as a factor if and only if the remainder is zero; $f(x)$ has $(x - r)$ as a factor if and only if $f(r) = 0$.

1. Haley and Lillian each prove that $(x - 7)$ is a factor of the polynomial $f(x) = x^3 - 10x^2 + 11x + 70$.

Haley

$(x^3 - 10x^2 + 11x + 70) \div (x - 7)$

$$
\begin{array}{r|l}
 & x^2 - 3x - 10 \\
x - 7 & x^3 - 10x^2 + 11x + 70 \\
 & \underline{x^3 - 7x^2} \\
 & -3x + 11x \\
 & \underline{-3x + 21x} \\
 & -10x \quad 70 \\
 & \underline{-10x \quad 70} \\
 & 0
\end{array}
$$

Lillian

$f(x) = x^3 - 10x^2 + 11x + 70$

$f(7) = 7^3 - 10(7)^2 + 11(7) + 70$

$f(7) = 343 - 490 + 77 + 70$

$f(7) = 0$

Explain why each student's method is correct.

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You can continue to factor the polynomial $f(x) = x^3 - 10x^2 + 11x + 70$.

From Haley and Lillian's work, you know that $f(x) = (x - 7)(x^2 - 3x - 10)$.

The quadratic expression can also be factored.

$$f(x) = (x - 7)(x^2 - 3x - 10)$$

$$f(x) = (x - 7)(x + 2)(x - 5)$$

2. Use the Factor Theorem to prove each factor shown in the worked example is correct.

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3. What other method(s) could you use to verify that the factors shown in the worked example are correct?

4. Use the Factor Theorem to prove that $f(x) = (x + 1 - 3i)(x + 1 + 3i)$ is the factored form of $f(x) = x^2 + 2x + 10$.

5. Determine the unknown coefficient, a, in each function.

a. $f(x) = 2x^4 + x^3 - 14x^2 - ax - 6$ if $(x - 3)$ is a linear factor.

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b. $f(x) = ax^4 + 25x^3 + 21x^2 - x - 3$ if $(x + 1)$ is a linear factor.

Talk the Talk

Given the information:

$$p(x) = x^3 + 6x^2 + 11x + 6, \text{ and}$$
$$p(x) \div (x + 4) = x^2 + 2x + 3 \;\; \text{R} -6$$

Determine whether each statement is true or false. Explain your reasoning.

1. $p(-4) = 6$

2. $p(x) = (x + 4)(x^2 + 2x + 3) - 6$

3. -4 is not a zero of $p(x)$

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4. -2 is a zero of $p(x)$

Be prepared to share your solutions and methods.